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On the homoclinic orbit for convection in a fluid layer heated from below

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Abstract

The homoclinic orbit associated with the transition from steady convection to Chaos in the weak turbulent regime is difficult to recover in simple computational procedures because of its unstable nature. A simple procedure which recovers the homoclinic orbit up to the desired accuracy is presented. This procedure combines the insight obtained from previous work on a local non-linear analysis of the problem and the Adomian's decomposition method of solution which provides a global semi-analytical solution. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Transitions from steady convection to Chaos in a fluid layer heated from below are typically associated with a homoclinic explosion when the trajectory which originally wanders around one fixed point moves away towards the other fixed point. The trajectory's behaviour depends on the Rayleigh number and on the initial conditions. While the existence of a homoclinic orbit in the convective flow was already established, the difficulty to recover computationally this orbit is documented as well [1]. A simple procedure allowing the computational recovery of the homoclinic orbit is presented in this paper by using Adomian's decomposition method [2,3], which provides in principle an analytical solution, in conjunction with previous analytical results which provide effective guidelines. The detailed analysis of the problem under consideration was presented by Vadasz [4], and for the corresponding problem of gravity driven or centrifugally induced convection in a porous layer, by Vadasz and Olek [5] and Vadasz and Olek [6], respectively. The objective of the present paper is to present results regarding the salient aspects of the computational recovery of the homoclinic orbit, which have not been presented or discussed elsewhere.

2. Problem formulation and method of solution

The problem consists of an infinite fluid layer subject to gravity and heated from below which is subject to stress free boundary conditions on the horizontal boundaries. For convective rolls having axes parallel to the shorter dimension (i.e. y) $v = 0$, and the governing equations can be presented in terms of a stream function defined by $u = -\partial\psi/\partial z$ and $w = \partial\psi/\partial x$, which yields the following system of partial differential equations presented in a dimensionless form

$$\left[\frac{1}{Pr}\left(\frac{\partial}{\partial \hat{t}} + \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\right) - \nabla^2\right]\nabla^2\psi = Ra\frac{\partial T}{\partial x} \quad (1)$$

$$\frac{\partial T}{\partial \hat{t}} - \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \quad (2)$$

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where the two-dimensional Laplacian operator is defined in the form $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial z^2$ and the boundary conditions for the stream function are $\psi = \partial\psi/\partial z = 0$ on the horizontal boundaries. Here u, v and w are the three components of the velocity vector, T is temperature, x and y are the horizontal space coordinates, z is the vertical space coordinate and $\hat{t}$ is time. The dimensionless groups appearing in Eqs. (1) and (2) are the Rayleigh number defined as $Ra = \beta_* \Delta T_* g_* h_*^3/\alpha_* \nu_*$ and the Prandtl number defined as $Pr = \nu_*/\alpha_*$, where β_* is the thermal expansion coefficient, ΔT_* is the constant temperature difference between the hot bottom and cold top boundaries, g_* is the gravity acceleration, h_* is the height of the layer, ν_* and α_* are the kinematic viscosity and the thermal diffusivity of the fluid, respectively.

To obtain the complete solution to the non-linear coupled system of partial differential Eqs. (1) and (2) we represent the stream function and temperature in the form

$$\psi = 3\sqrt{\lambda(R-1)}X(\hat{t})\ \sin(\kappa x)\ \sin(\pi z) \tag{3}$$

$$T = 1 - z + \frac{\sqrt{2\lambda(R-1)}}{\pi R} Y(\hat{t})\ \cos(\kappa x)\ \sin(\pi z) + \frac{(R-1)}{\pi R} Z(\hat{t})\ \sin(2\pi z) \tag{4}$$

where $R = Ra/Ra_c$ is the scaled Rayleigh number, which upon using the wavenumber corresponding to the convection threshold, i.e. $\kappa_{cr} = \pi/\sqrt{2}$, yields for the critical Rayleigh number $Ra_c = 27\pi^4/4$, and $\lambda = 8/3$. This representation is equivalent to a Galerkin expansion of the solution in both x and z directions, truncated when $i + j = 2$, where i is the Galerkin summation index in the x direction and j is the Galerkin summation index in the z direction. Substituting (3) and (4) into the Eqs. (1) and (2), multiplying the equations by the orthogonal eigenfunctions corresponding to (3) and (4) and integrating them over the height of the domain and over the wavelength of the convection cell in the vertical and horizontal directions, respectively, i.e.

$$\int_0^{\pi/\kappa} \mathrm{d}x \int_0^1 \mathrm{d}z(\cdot),$$

yields a set of three ordinary differential equations for the time evolution of the amplitudes in the form

$$\dot{X} = Pr(Y - X) \tag{5}$$

$$\dot{Y} = RX - Y - (R-1)XZ \tag{6}$$

$$\dot{Z} = \lambda(XY - Z) \tag{7}$$

where the time was rescaled in the form $t = 3\pi^2\hat{t}/2$ and the dots ($\dot{}$) denote time derivatives d( )/dt. Eqs. (5)–(7) are the famous Lorenz equations [7,1], which are satisfied by the motionless solution $X = Y = Z = 0$ that is stable when $R < 1$, by the steady convective solutions $X = Y = \pm 1$ and $Z = 1$ which are stable when $1 < R < R_{c2}$, and by Chaotic or periodic solutions for values of $R > R_{c2}$. The transition from the steady to the Chaotic solution occurs via a subcritical Hopf bifurcation [1,4] and is associated with a homoclinic explosion when the trajectory which originally wanders around one steady convective solution (fixed point) moves away towards the other fixed point.

Adomian's decomposition method [2,3,8] was adopted in solving the system (5)–(7) to obtain an analytical solution in terms of infinite power series. The practical need to evaluate the solution and obtain numerical values from the infinite power series, the consequent series truncation, and the practical procedure to accomplish this task, transform the analytical results into a computational solution evaluated up to a finite accuracy. Details regarding the method of solution as applicable to solving the system (5)–(7) are presented by Vadasz [4] and, Vadasz and Olek [5,6].

3. Results and discussion

The results of $X(t)$, $Y(t)$ and $Z(t)$ are presented in terms of projections of the trajectory's data points onto the planes $Y = 0$, $Z = 0$ and $X = 0$. As long as the initial conditions are not too far away from one of the fixed points $X = Y = Z = 1$ or $X = Y = -1$, $Z = 1$ it is relatively easy to recover the orbit associated with the Hopf bifurcation relevant to the transition from steady convection to Chaos. However as the initial conditions depart significantly from the fixed points the orbit becomes more and more unstable and its recovery becomes more difficult. The region around the origin poses particular difficulties because (i) it is far away from both fixed points, and (ii) the Z axis being part of the stable manifold of the origin prevents the choice of initial conditions on this axis, as the solution then will naturally converge towards the origin ($X = Y = Z = 0$) and prevent the recovery of the homoclinic orbit. The method adopted here is the use of quantitative approximations based on Vadasz [4] and successive computations for initial conditions around the origin but not necessarily too close to it. This procedure allowed to establish that the homoclinic orbit in the neighbourhood of the origin lies on the plane $Y \approx 1.83333\ X$. Accordingly we chose the initial conditions as close as possible to the origin on this plane. The results presented here correspond to the initial con-

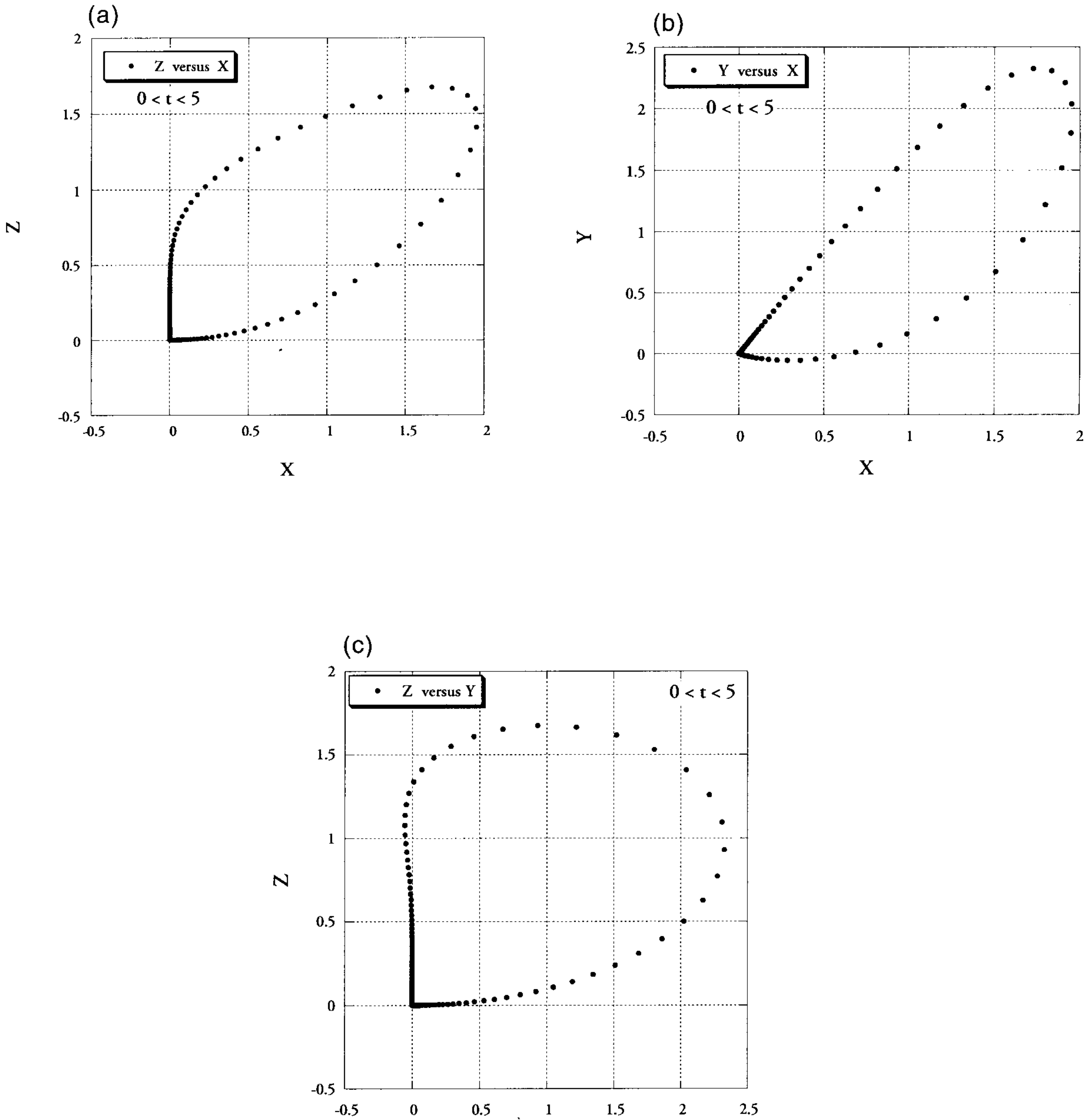

Fig. 1. The one-sided homoclinic orbit (trajectory's data points are not connected); (a) projection of trajectory's data points on the plane $Y=0$, (b) projection of trajectory's data points on the plane $Z=0$, (c) projection of trajectory's data points on the plane $X=0$.

ditions $X_0=10^{-6}$, $Y_0=1.83333\times10^{-6}$ and $Z_0=0$. Then for $R=13.926557407520$ we obtain the one-sided (single branch) homoclinic orbit as presented in Fig. 1, observed when the trajectory makes one single loop around the fixed point (i.e. for $0\leq t\leq 5$). The accuracy (number of significant digits) required for the value of R in order to recover the periodic orbit far away from both fixed points was analysed and discussed by Vadasz [4]. There is not much novelty in recovering the one-sided homoclinic orbit; this has been done previously and reported by Sparrow [1].

The interesting part is the ability to use this procedure to recover the complete two-sided (both branches) homoclinic orbit. This is accomplished by a

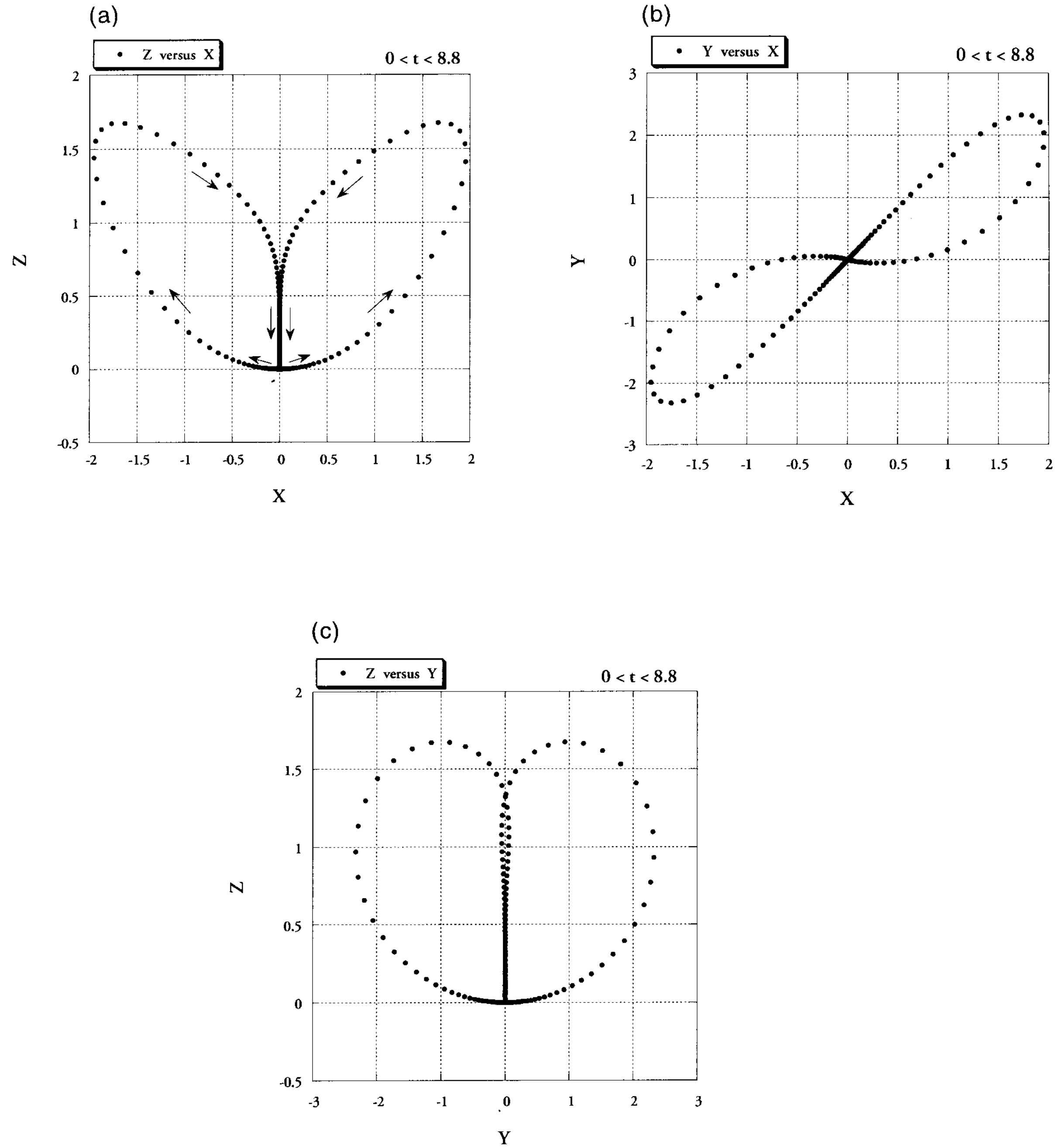

Fig. 2. The two-sided homoclinic orbit (trajectory's data points are not connected); (a) projection of trajectory's data points on the plane $Y=0$, (b) projection of trajectory's data points on the plane $Z=0$, (c) projection of trajectory's data points on the plane $X=0$.

very slight variation of R. For $R=13.926557407521$ we obtain the complete homoclinic orbit as presented in Fig. 2 following the trajectory to make a complete loop around both fixed points. The direction of the trajectory is marked by arrows on Fig. 2(a), representing the projection of the trajectory's data points onto the plane $Y=0$. It is evident from the figure that the trajectory starting from the initial conditions at $Z_0=0$ and $X_0=10^{-6}$ (and $Y_0=1.83333\times 10^{-6}$ not observed on this projection) moves counter-clockwise on the right branch of the homoclinic orbit and returns straight to the origin from above. At the origin the trajectory makes a sudden right turn switching to the left branch of the orbit where it moves clockwise making a complete loop by returning to the origin from above (actually very close to it but not exactly to the origin). Fig. 2(b), (c) represent the projection of the same trajectory on the planes $Z=0$ and $X=0$, respectively. The closer we choose the value of X_0 to the origin, the closer the trajectory will recover the accurate homoclinic orbit (recall that the value of Y_0 is selected to lie on the plane $Y \approx 1.83333\ X$). Naturally, we cannot choose the origin itself as the initial conditions, because it lies on the Z axis which being a part of the stable manifold of the origin, will cause the solution to remain at the origin, recovering the trivial solution.

4. Conclusions

A simple procedure for computational recovery of the homoclinic orbit for the problem of thermal convection in a fluid layer heated from below was presented. The procedure combining previous results obtained via the weak non-linear analysis and Adomian's decomposition method of solution yield the complete two-sided homoclinic orbit up to the desired accuracy.

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